Practice Final Exam, Physics 115A
Fall 2018

This exam is closed book, closed notes, closed calculators/phones. Please show your work for full credit. You may make free use of anything on the "Useful Formulae" page. There are eight problems, with point values indicated below out of a total of 80 points.

You have 180 minutes.

Problem	Points
1	9
2	10
3	6
4	5
5	10
6	10
7	14
8	16
Total	80

Useful formulae

Integration by parts: $\int_a^b f(x)\frac{\partial g(x)}{\partial x} = -\int_a^b \frac{\partial f(x)}{\partial x} g(x) + f(x)g(x)|_a^b$
Commutators: $[A,B] \equiv AB - BA$.
Gaussian Integrals: $\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\pi/\lambda}$. Note also $\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx = -\frac{d}{d\lambda}\int_{-\infty}^{\infty} e^{-\lambda x^2} dx$.
Super helpful completing the square: $\int_{-\infty}^{\infty} e^{-(Ax^2+Bx)} dx = \sqrt{\frac{\pi}{A}} e^{B^2/4A}$
Fourier transform conventions: $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$, $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\, dk$
Probability: Prob($a \leq x \leq b$ at time t) $= \int_a^b \Psi^*(x,t)\Psi(x,t)dx$
Probability Current: $\frac{\partial}{\partial t}|\Psi|^2 = -\frac{\partial}{\partial x}J(x,t)$ for $J(x,t) = -\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)$
Position and Momentum Operators: $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ (position space); $\hat{x} = i\hbar\frac{\partial}{\partial p}$ and $\hat{p} = p$ (momentum space); canonical commutation relation $[\hat{x},\hat{p}] = i\hbar$ (any space)
Expectation Values: $\langle \hat{Q}(x,p)\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\hat{Q}(x,-i\hbar\frac{\partial}{\partial x})\Psi(x,t)dx$ (position space); $\langle\hat{Q}\rangle = \langle\Psi|Q|\Psi\rangle$ (any space)
Uncertainties: $\sigma_Q \equiv \sqrt{\langle Q^2\rangle - \langle Q\rangle^2}$ for any operator Q
Schrödinger Equation: $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\Psi(x,t)$ (position space); $i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$ (any space)
Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ (any space)
Time-independent Schrödinger Equation: $\hat{H}\psi(x) = E\psi(x)$ for separable solutions $\psi(x,t) = e^{-iEt/\hbar}\psi(x)$.
Orthonormality and completeness: $\int \psi_m^*\psi_n dx = \delta_{mn}$ and $\Psi(x,0) = \sum_n c_n\psi_n(x)$ with $c_n = \int \psi_n^*\Psi(x,0)dx$.
Infinite square well: For $\hat{H} = \frac{\hat{p}^2}{2m} + V$ with $V = 0$ for $0 \leq x \leq a$, ∞ otherwise. Stationary solutions $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x/a)$, $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$
Harmonic oscillator: For $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, raising/lowering operators $a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} \mp i\hat{p})$, $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$, $\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$, $[a_-, a_+] = 1$, $\hat{H} = \hbar\omega(a_+a_- + 1/2)$, $E_n = \hbar\omega(n+1/2)$, $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$, $a_-\psi_n = \sqrt{n}\psi_{n-1}$, $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$.
Free particle: For $\hat{H} = \frac{\hat{p}^2}{2m}$. Plane waves $\psi(x,t) = Ae^{i(kx-\omega t)}$, $k = p/\hbar = \sqrt{2mE}/\hbar$, $E = \hbar\omega = \frac{\hbar^2k^2}{2m}$, Wave packets $\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int\phi(k)e^{ikx}dk$ and $\phi(k) = \frac{1}{\sqrt{2\pi}}\int\Psi(x,0)e^{-ikx}dx$
Scattering problems: For $E > 0$, incoming plane waves from $x = -\infty$, Ae^{ikx} $(k > 0)$, reflected waves returning to $x = -\infty$, Be^{-ikx}, and transmitted waves going to $x = +\infty$, Fe^{ikx}. Match in regions with nonzero potential.
Dirac well: For $V(x) = -\alpha\delta(x)$ $(\alpha > 0)$, one bound state $\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}$ with energy $E = -\frac{m\alpha^2}{2\hbar^2}$, plus scattering states with $E > 0$.
Generalized Uncertainty Principle: $\sigma_A\sigma_B \geq \left|\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right|$
Time Evolution of Expectation Values: $\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$

1. [9 points total] At $t = 0$, a particle of mass m in an infinite square well with walls at $x = 0, a$ is in a superposition of the (normalized) first and third stationary states

$$\Psi(x,0) = A(2\psi_1(x) - i\psi_3(x))$$

   (a) [2 points] What must A be in order for Ψ to be a normalized wavefunction?

   (b) [3 points] What is the probability that the particle is in the left half of the box (i.e., between $x = 0$ and $x = a/2$) at $t = 0$? *Hint: Instead of trying to do a complicated integral directly, notice that for odd n, $\psi_n(x) = \psi_n(a - x)$ for $0 \le x \le a$. You can use this to relate $\int_0^{a/2} \psi_m^* \psi_n dx$ to $\int_0^a \psi_m^* \psi_n dx$ when m, n are both odd.*

   (c) [1 point] Given the initial wavefunction $\Psi(x, 0)$, what is the probability of measuring the energy to be $E = \frac{2\pi^2\hbar^2}{ma^2}$ at $t = 0$?

   (d) [3 points] What is the probability that the particle in the above state $\Psi(x, 0)$ at $t = 0$ is in the left half of the box at some later time t?

2. [10 points total] Consider an observable A with corresponding operator $\hat{A}$, with three normalized eigenstates $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ whose eigenvalues are a_1, a_2, a_3.

   (a) [2 points] You prepare a particle in a state $|\Psi\rangle = \frac{2}{3}|\psi_1\rangle + \frac{1}{3}|\psi_2\rangle - \frac{2}{3}|\psi_3\rangle$. If you were to measure A for this particle, what are the possible outcomes, and what are their probabilities?

   (b) [2 points] Suppose you know that $[\hat{A}, \hat{H}] = i\hbar$. What is $\frac{d}{dt}\langle A\rangle$ for a particle in the state $|\Psi\rangle$? You may assume the operator $\hat{A}$ is itself independent of time.

   (c) [1 point] Suppose you go ahead and measure A and find the value a_3. What is the state of the particle immediately after this measurement?

   (d) [2 points] If you measure A again immediately after your first measurement (the one that yielded a_3), what are the possible outcomes, and what are their probabilities?

   (e) [3 points] You prepare a particle in the same initial state as before, $|\Psi\rangle = \frac{2}{3}|\psi_1\rangle + \frac{1}{3}|\psi_2\rangle - \frac{2}{3}|\psi_3\rangle$. If you were to make a measurement of the observable corresponding to the operator $\hat{A}^{-1}$ on this state (i.e., the inverse of $\hat{A}$), what are the possible outcomes, and what are their probabilities?

3. [6 points total] Let's derive some results for the quantum version of the *virial theorem.*

   (a) [4 points] Use your knowledge of the time evolution of expectation values to show that
   $$\frac{d}{dt}\langle \hat{x}\hat{p}\rangle = 2\left\langle \frac{\hat{p}^2}{2m}\right\rangle - \left\langle \hat{x}\frac{\partial V}{\partial \hat{x}}\right\rangle$$
   (The operators $\hat{x}$ and $\hat{p}$ are themselves independent of time.)

   (b) [1 point] Show that for stationary states this reduces to the quantum version of the virial theorem,
   $$2\left\langle \frac{\hat{p}^2}{2m}\right\rangle = \left\langle \hat{x}\frac{dV}{d\hat{x}}\right\rangle$$

   (c) [1 point] Prove that this implies $\left\langle \frac{\hat{p}^2}{2m}\right\rangle = \langle V\rangle$ for stationary states of the harmonic oscillator.

4. [5 points total] Consider a particle of mass m subject to an attractive delta function potential well $V(x) = -\alpha\delta(x)$, where $\alpha > 0$. You've carefully prepared it in the bound state of this potential. At the time $t = 0$ your friend bumps your lab bench, which causes the strength of your delta function potential to instantaneously change to $V(x) = -\gamma\delta(x)$ for some other constant $\gamma > 0$. This happens so fast that the wavefunction of the particle is unchanged at the moment of the change. What is the probability that the particle remains in a bound state of the Hamiltonian?

5. [10 points total] Short answers (at most a few sentences each + math where needed)

   (a) [2 points] Show that the expectation value of an anti-hermitian operator (i.e. an operator $\hat{Q}$ for which $\hat{Q}^\dagger = -\hat{Q}$) is imaginary.

   (b) [2 points] You have a particle in the state $|\Psi\rangle$ and measure an observable Q corresponding to a hermitian operator $\hat{Q}$. If $\hat{Q}$ has a discrete spectrum of eigenvectors $|\psi_n\rangle$, describe what "collapse of the wavefunction" means in terms of $|\Psi\rangle$ and $|\psi_n\rangle$. What does "collapse of the wavefunction" mean if $\hat{Q}$ has a continuous spectrum of eigenvectors?

   (c) [2 points] Why do we require that physical states live in Hilbert space?

   (d) [2 points] How are states $|\Psi\rangle$, position-space wavefunctions $\Psi(x)$, and momentum-space wavefunctions $\Psi(k)$ related? Use Dirac notation.

   (e) [2 points] If the Hamiltonian for a two-level system is written with respect to some basis as a matrix
   $$\mathbf{H} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$
   for real constants a, b, what do you know about the basis vectors?

6. [10 points total] Consider a particle of mass m in a harmonic oscillator potential with frequency ω.

   (a) [2 points] The ground state is defined by $a_-|\psi_0\rangle = 0$. Using Dirac notation, show that this leads to a differential equation satisfied by the position-space ground state wavefunction $\psi_0(x)$. You will want to use the matrix elements of $\hat{x}$ and $\hat{p}$ in position space, namely $\langle x|\hat{x}|x'\rangle = x'\delta(x-x')$ and $\langle x|\hat{p}|x'\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x')$.

   (b) [2 points] Solve this differential equation to find the explicit form of the (properly-normalized) position-space ground state wavefunction $\psi_0(x)$.

   (c) [2 points] Using Dirac notation, show that $a_-|\psi_0\rangle = 0$ leads to a differential equation satisfied by the momentum-space ground state wavefunction $\psi_0(k)$. In analogy with part (a), you will want to use the matrix elements of $\hat{x}$ and $\hat{p}$ in momentum space, namely $\langle p|\hat{p}|p'\rangle = p'\delta(p-p')$ and $\langle p|\hat{x}|p'\rangle = i\hbar\frac{\partial}{\partial p}\delta(p'-p)$.

   (d) [2 points] Solve this differential equation to find the explicit form of the (properly-normalized) momentum-space ground state wavefunction $\psi_0(k)$.

   (e) [2 points] Compute the Fourier transform of $\psi_0(x)$ and show that it agrees with your answer from part (d).

7. [14 points total] Consider the finite square well with $V = -V_0$ for $-a \leq x \leq a$ and $V = 0$ for $|x| > a$.

   (a) [1 point] Why can the separable solutions to the time-independent Schrödinger equation be expressed in terms of even and odd functions of x?

   (b) [3 points] Write down the general form of the *even* bound-state wavefunctions in the three regions $x < -a$, $-a \leq x \leq a$, and $x > a$.

   (c) [3 points] Find the transcendental equation for the allowed energies. You do not need to solve it.

   (d) [3 points] Now consider the scattering states with energies $E > 0$. Write down the general form of the scattering solutions in the three regions $x < -a$, $-a \leq x \leq a$, and $x > a$.

   (e) [4 points] Write down the matching conditions at each of the boundaries between these regions. You do not need to solve them.

8. [16 points total] Let's end with four short problems involving the most important potential in the universe, the quantum harmonic oscillator.

(a) [4 points] A particle is in the ground state of the harmonic oscillator with frequency ω, when suddenly the spring constant doubles so that $\omega_{new} = 2\omega$, without initially changing the wave function. What is the probability that a measurement of the energy would still return the value of $\hbar\omega/2$? What is the probability of getting $\hbar\omega$?

(b) [5 points] Consider a particle in the ground state of a harmonic oscillator of frequency ω. Compute $\langle x \rangle, \langle x^2 \rangle, \langle p \rangle, \langle p^2 \rangle$, and use these to verify that the ground state saturates the Heisenberg uncertainty principle. Hint: you can find $\langle x \rangle$ and $\langle p \rangle$ quickly by thinking about symmetries.

(c) [4 points] Recall the *coherent states* of the harmonic oscillator, which are eigenfunctions of the lowering operator a_-. These states $|\alpha\rangle$ satisfy $a_-|\alpha\rangle = \alpha|\alpha\rangle$ with a complex eigenvalue α. In position space, this implies a differential equation satisfied by the position-space wavefunction of the coherent state, $S_\alpha(x)$. Write down this differential equation and show that it is satisfied by *travelling gaussian wavepackets*, i.e. wavefunctions of the form $S_\alpha(x) = Ae^{-ax^2-bx}$. Determine the constants A, a, and b for a properly-normalized coherent state of eigenvalue α in terms of $m, \omega, \hbar$, and α.

(d) [3 points] Find the allowed energies of the *half* harmonic oscillator,

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & \text{for } x > 0 \\ \infty & \text{for } x < 0 \end{cases}$$

This represents, for example, a spring that can be stretched from its equilibrium position but not compressed. This problem can be solved with some careful thought but very little calculation – just be sure to justify your answer.

Congratulations, you've reached the end of the exam.