# Conformal Mapping in Origami: From Life to the Stars

Milo Brown
*Department of Physics, University of California, Santa Barbara, CA 93106*
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Although origami began as a ceremonial Japanese art form, this ancient art of folding has widespread modern applications in fields ranging from biomedical engineering to the space industry. These powerful uses drive research in mathematical methods, such as conformal mapping, to design origami structures. Conformal mapping transforms the complex plane, simplifying complicated shapes while preserving the angles between any two curves. In the context of origami, the conformal mapping method inverts the output of a conformal map to design origami crease patterns. The conformal mapping technique can efficiently design foldable conical structures by inverting the complex logarithm.

## I. INTRODUCTION

Origami is the art of folding paper into three-dimensional figures. Origami is a Japanese term, originating from the Japanese words ori (fold) and kami (paper) [1]. However, the geographical origin of origami is often debated because paper does not preserve well over long periods of time, leaving less material evidence for archaeologists. Some suggest that Buddhist monks brought Zhezhi, the Chinese art of paper folding, to Japan for ceremonial purposes [2]. Others assert that the Japanese invented origami about a thousand years ago. Long after the proposed origin of origami, Ihara Saikaku wrote the first recorded reference to origami in a poem that described folded butterflies, similar to Fig. 1, in a Shinto wedding [3]. This poem reveals that origami was a common cultural phenomenon in Japan by the late 17th century. Historically, origami has been used for ceremonial and recreational purposes. However, origami also has modern applications in science, mathematics, engineering, and technology.

Although origami initially developed as a Japanese art form, scientists worldwide apply the principles of origami to allow structures to compactly traverse long distances and reassemble with little effort. Origami becomes essential when structures are inaccessible at their destination, such as the equipment involved in the space industry. Aerospace engineers must compactly deploy and reassemble massive structures such as solar sails and parabolic antennas using principles of origami at the target location [4] [5]. Additionally, architects and civil engineers use origami principles to determine structural efficiency of foldable surfaces such as temporary structures or convertible roofs [6]. Engineers and mechanics also apply origami in vehicle design to protect passengers from injury from a crash by optimizing the folding of the car [7]. Bioengineers turn to DNA origami to assemble biomolecules at the nanoscale, and medical engineers use origami to compact and deploy stent grafts to assist minimally invasive surgery [8] [9]. Origami's multifaceted potential drives the need to standardize and simplify the origami design process using mathematical methods.

FIG. 1. The above depicts a modern origami butterfly [10]. Saikaku's poem describes paper butterflies at a Shinto wedding–the first written description of origami.

Engineers and scientists can use conformal mapping to design origami-based structures. Conformal mapping transforms a complex plane while preserving angles between any two curves in the plane. This mapping technique preserves solutions to Laplace's equation, allowing representation of fluid flow, heat flow, and electric fields; it also preserves Dirichlet and Neumann boundary conditions [11]. By applying principles of transformation, conformal mapping can simplify and unify origami design techniques.

## II. METHODS

### A. Conformal Mapping

Conformal mapping preserves angles between curves in a plane transformation that allows scientists to simplify the behavior of complicated fields to solve physical problems. For example, Fig. 2 transforms circular curves to flat lines using the complex logarithm. Analysis of the polar form in Eq. 1 reveals that in this conformal mapping example, the circular complex exponential transforms to a sum of scalar components, creating a flat output plane.

$$ln(z) = ln(re^{i\theta}) = ln(r) + ln(e^{i\theta}) = ln(r) + i\theta \quad (1)$$

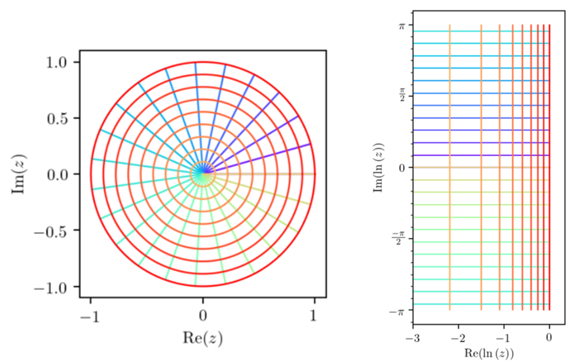


FIG. 2. This conformal map transforms circular curves to linear curves using the complex logarithm, which is the inverse of the complex exponential. This illustrates how conformal mapping can transform the plane and simplify the behavior of complicated curves. The figure was generated in python using open source conformal mapping documentation [12].

As illustrated in Eq. 1, complex functions have complex input z and complex output f(z). The complex output can be expressed as a real component, $\phi$(x,y), and an imaginary component, $\psi$(x,y), where x is the real component and y is the imaginary component of the complex input. Eq. 2 demonstrates this relationship between f(z) and $\phi$ and $\psi$, as well as the relationship between z and x and y.

$$f(z) = \phi + i\psi, \quad \text{where} \quad z = x + iy \quad (2)$$

A map from the x-y plane to the $\phi$-$\psi$ plane will be conformal at a point if it preserves angles between any 2 curves passing through that point. The complex output map is conformal at a point if it is analytic and if its derivative is not equal to zero at that point. An analytic function observes the Cauchy-Riemann conditions shown in Eq. 3.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad (3)$$

Equipotential lines occur when $\phi$ is constant and streamlines occur when $\psi$ is constant. Equipotential and streamlines are orthogonal, and can represent physical phenomena; for example, streamlines can represent the direction of fluid flow. Conformal mapping simplifies the movement of the streamlines by transforming them to an appropriate plane, allowing scientists to work with complicated fields with ease. Fig. 3 shows an illustration of the fluid flow example, comparing the analytic flow in Fig. 3a to the flow with circulation and with a source in 3b and 3c, dipole flow in 3d, and flow in a wedge in 3e. Conformal mapping can transform each of these complicated flow phenomena into the simpler flow in Fig. 3a.

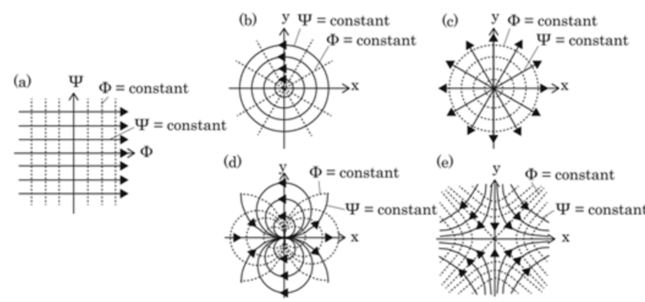


FIG. 3. The simple flow in (a) is the conformal representation of (b), (c), (d), and (e) using different transformation functions (f(z)). This figure can be found in Ishida et al. 2014 [13].

We can use the principles of conformal mapping to transform initial crease lines to fold lines in origami structures. This simplifies the complicated process of creating crease lines from the beginning.

## III. RESULTS

### A. Modeling Origami Using Conformal Mapping

To model origami structures, we can visualize fold lines as equipotential lines and streamlines and transform the fold lines to crease patterns, using the inverse analytic function to return from the $\phi$-$\psi$ plane to the x-y plane. This reverse-engineering technique simplifies the process of creating crease lines, which is difficult to do from scratch.

To create a physical origami structure that exists on a plane, the structure must be flat-foldable. In other words, the fold lines of the structure cannot be curved. Since transforming the lines would create a curved fold, the coordinates of the nodes are transformed as shown in Fig. 4, and a straight line connects the nodes to create a new fold line. Therefore, the angles between the new fold lines are not exactly preserved, thus the new fold plane is not exactly conformal [13].

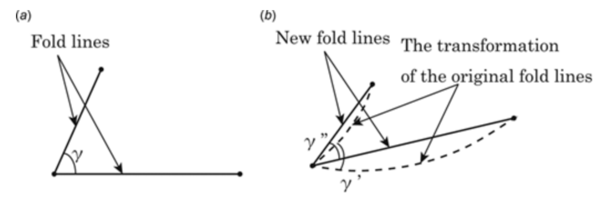


FIG. 4. The figure shows (a) the original fold lines and (b) the transformed fold lines. Notice that the original conformal transformation produces curved lines, and straight lines are retrieved by connecting the nodes. This figure originally appeared in Ishida et al. 2014 [13].

## B. Flat-Foldability

An origami structure's crease pattern illustrates where a fold occurs. The crease pattern can describe concave valley folds and convex mountain folds. Flat-foldable angles are bounded with a maximum angle limit of $\pi$ to $-\pi$, as demonstrated in Fig. 5. The structure is considered flat-foldable at a vertex when alternating angles at the vertex sum to 180 degrees and the number of mountain and valley folds differ by two [14]. Flat-foldability is particularly interesting to scientists and engineers because a flat-foldable structure becomes compact, allowing more robust applications. A flat-foldable structure must be flat-foldable at both its final folded configuration in the $\phi$-$\psi$ plane and its initial crease lines in the x-y plane [13].

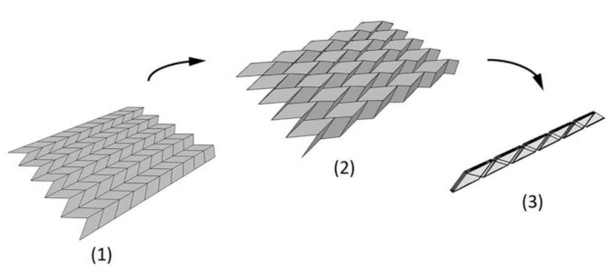


FIG. 5. This figure demonstrates flat-foldability in Miura-ori origami. (1) Shows the crease lines of the fully expanded structure. (2) Shows partially folded structure. (3) Shows the fully compacted flat-foldable structure. This figure originally appeared in Meloni et al. 2021 [14].

## C. Foldable Conical Structures

The conformal mapping technique is much more efficient at creating conical structures than previous techniques, which involved designing new crease patterns from scratch.

When we bend the crease lines of a circular flat-foldable structure in the same direction, we can fold a circular membrane such as in Fig. 6. This crease pattern is called a twist bucking pattern. The pattern involves 6 elements across the lateral direction [15]. We can represent this crease pattern as a complex map using Eq. 4, where k is a constant. Eq. 4 uses z to express the new folded plane and f(z) to express the original plane.

$$f(z) = i \cdot k \log(z) \tag{4}$$

Fig 6 contains visualizations of conical origami structures. The left section depicts the crease patterns, the middle section shows the crease patterns after angle correction, and the right section depicts the physical models of each conical design. The $fsolve$ function in MatLab is used to tune the $\beta$ variable in Eq. 6 by minimizing the difference of the nonlinear equation – this corrects the angle and allows the cone to close [13].

$$2N\beta' + \Phi - 2D\alpha'' = 2\pi \tag{5}$$

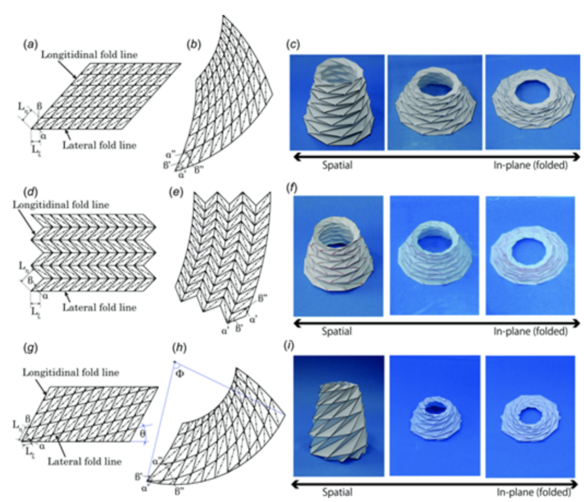


FIG. 6. The left subfigures (a), (d), (g) show crease patterns. The middle subfigures (b), (e), (h) show angle-corrected crease patterns. The right subfigures (c), (f), (i) show physical models. The figure is from Ishida et al. 2014 [13].

The example of foldable conical structures demonstrates the potential of the conformal mapping method of origami.

## IV. DISCUSSION

The conformal mapping technique provides a more efficient method of designing origami crease patterns. Although this paper explores foldable conical structures with six fold lines, it could have easily expanded to flat-foldable structures with four fold lines, such as the Miura-ori origami pattern depicted in Fig. 5. The engineering and materials resources on origami and conformal mapping are fairly robust. However, far fewer authors explore the artistic possibilities of the conformal mapping technique. Conformal mapping has the potential to provide artists with a powerful design tool, allowing them to create more complicated structures.

The conformal mapping technique is limited because it can produce structures that are unfoldable. It is also somewhat inconvenient for producing some flat-foldable structures because angles must be corrected at each node [13]. Other forms of computational origami, which uses algorithms to design folding structures, may provide more effective designs for certain structures to artists and scientists alike.

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